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OPTIMAL TARIFFS AND INTERNATIONAL TRADE

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Abstract

Optimal tariff strategies, with and without optimal responses, are derived, based on a linear partial equilibrium trade model, A, and a nonlinear general equilibrium trade model with two nations, B. Under free trade, models A and B are applied to prove that it is always profitable for one nation, N_1, to introduce a strictly positive tariff on imports, T_1, if other nations do not introduce tariffs. With model B, it is proved that it is rational also for nation N_2, to introduce a tariff, T_2, if N_1, introduces a tariff, T_1. The Nash equilibrium tariff combination is unique and stable. The economic results, in both nations, are however higher with free trade, than in the Nash equilibrium tariff solution. If both nations know that the other nations will respond to increasing tariffs, and select the optimal tariff, then it is not rational for any nation to leave free trade and introduce tariffs. Free trade agreements are rational for the participants. It is also shown that it may be rational for a dictator, to introduce a tariff, even if this is not rational for the consumers in the same nation. The dictator may keep the tariff revenues and use these for other purposes. In such a case, the tariff reduces the consumer surpluses in both nations.

Keywords: Optimal tariffs, international trade, Nash equilibrium, Free trade, Dynamic non- zero-sum game theory.

1. Introduction

1.1. General International Economics

Krugman and Obstfeld (2000) is a famous book that covers the field international economics, theory and policy. One of the interesting parts in this book contains a section where they show that the net welfare effect of a tariff is a strictly concave and quadratic function of the size of the tariff. For sufficiently small tariffs, the function is strictly positive. This means that the optimal tariff is strictly positive. In other words, free trade is not the optimal solution according to that analysis. Krugman obtained the Nobel Memorial Prize in Economic Sciences in 2008. In this paper, it will also be demonstrated that the optimal tariff is strictly positive, in case the other nations do not introduce tariffs. The present analysis will also show that there is a stable Nash equilibrium with positive tariffs, but that the nations have an economically better situation if they cooperate and select free trade. In other words, free trade is a better solution for the participants than a world with tariffs. The optimal tariff problem is not a new topic. One classical approach is to derive a static formula that gives the optimal tariff as a simple function of the export supply elasticity. Dávila et al. (2025) give a good description of the earlier literature on the subject, and give an interesting description of assumptions behind the optimal tariff formulas from the past. They generalize a static optimal tariff formula to a dynamic version, where changing export supply elasticities can be handled.

1.2. A Deeper Focus on Tariffs

In this analysis, the topic will be extended even further. As Dávila et al. (2025) mention, for most functions, export supply elasticities are usually not constant when the export volume changes. Hence, a new optimal tariff formula is developed, which is not a function of the elasticities, but a function of the parameters of the supply and demand functions. This function also makes it possible to derive general results concerning the optimal tariff and several other things. Chacholiades (1978) gives a very good presentation of the many general theories of international trade and policy. He also touches several important issues connected to tariffs, such as the fact that several countries may introduce tariffs. He calls that retaliation. Chacholiades (1978) does not analyze the dynamic retaliation game further but suggests that if one nation introduces a tariff, other nations will be tempted to make tariff changes in, as he says "an infinite regression". He also asks the question if a tariff equilibrium exists, where the different nations would not be interested to change the tariffs any more. In this paper, the questions by Chacholiades (1978) are answered. The optimal dynamic tariff adjustment game is solved and the Nash equilibrium is presented. Furthermore, it is found that the Nash equilibrium, which is the result of individual and independent adjustments of the tariffs, is a stable equilibrium, but that the economic situations of the different nations in this equilibrium are worse than in the free trade solution, completely without tariffs. It is shown that the different nations, when they realize this fact, have a strong reason to negotiate tariff reductions until they reach the free trade solution, where the involved nations are better off. Relevant theories should be tested with relevant data. Empirical data of high quality and relevance to empirical estimations of effects of tariffs are often difficult to find. Furthermore, dynamics with asymmetric effects over time may sometimes be observed. Irwin (2014) describes this field and Bowden (2015) gives updated comments on the same topic. Broda el al. (2008) however find that market power explains tariff variation very well. Prior to World Trade Organization membership, countries set import tariffs 9 percentage points higher on inelastically supplied imports relative to those supplied elastically.

1.3. Tariffs and Game Theory

When a nation introduces a tariff, this may be interpreted as a form of economic warfare, namely an attack. The objective is to gain some economic advantage that should benefit the attacker and most likely harm the attacked nation. If also the attacked nation introduces a tariff, that may be considered as defense, with the objective to improve the economic situation of the attacked nation, and most likely also hurt the attacker. In the literature, the defense is often called retaliation. These changes of tariffs may continue in several steps. Lohmander (2018) presents general approaches to dynamic interactions and games of the different kinds of relevance in conflict situations. Game theory is central to the understanding of rational tariff decisions. If we initially have free trade and one nation introduces a tariff, it is reasonable to expect that the other nations will also consider the options to introduce tariffs. This may be considered and analyzed as a dynamic tariff game. Nash (1950) and Nash (1951) develop the general theory of noncooperative games and equilibria. Rasmusen (1990) writes a very instructive

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book on applications of the game theory concepts in many different and typical applications. Isaacs (1965) defines the concept differential games and applies that to solve many typical dynamic game problems, mostly from the military and mostly in continuous time. In the continuous time solutions, it is assumed that the games are zero sum games, which means that one player has the same objective function as the other player, but that one player wants to maximize and the other player wants to minimize the same objective function. Washburn (2003) continues in the same tradition, and develops zero sum game theory with applications to military battles, also including random strategies. The Washburn examples mostly concern problems from the navy and air force. Lohmander (2019a) and Lohmander (2019b) continue in the field of Washburn (2003) but primarily focus on and solve game problems in the army. Washburn and Kress (2009) cover a wide area of military operations analysis including zero sum games. There are however strong reasons to admit that many games of conflict cannot be considered as zero sum games. This is obvious in wars as Lohmander (2023) describes, where large areas, cities and infrastructure are destroyed and people are killed. In such cases, zero sum game theory is simply not relevant. For this reason, the final parts of the analyses in this paper are performed with dynamic non-zero-sum game theory. Lubik (2025) gives the general recommendation that tariffs should not be introduced. He advocates international cooperation. He states that tariffs distort markets, make resource allocation inefficient and often lead to destructive trade conflicts and trade wars. In 2025, however, President Trump in USA did not follow the advice from Lubik. Chu et al. (2025) describes the tariffs introduced by USA during 2025. These tariffs strongly influence world trade and the general economic situation in many parts of the world. The present analysis may be viewed as a theoretical structure that can be helpful when these empirical observations are interpreted. The following three sections focus on these topics: Section 2: The optimal tariff in partial equilibrium; Section 3: General equilibrium analysis; Section 4: The optimal tariffs without and with cooperation in general equilibrium.

2. The optimal tariff in partial equilibrium

First, the free trade solution in an importing nation, N_1 is derived, based on a linear demand function and a linear supply function from the world market. The solution is based on partial equilibrium analysis. Then, consumer surplus maximization, is used to derive the optimal import volume. It is shown that it is always optimal to import less than according to the free trade solution. When the optimal import volume is selected, the marginal consumer surplus is equal to the marginal cost, which means that the marginal consumer surplus, which equals the price level in N_1, exceeds the import price. The optimal import volume can be obtained in different ways. One option is to let the government introduces a tariff that corresponds to the difference between the optimal price in N_1 and the import price. Then, in some way, the government transfers the tariff revenue to the consumers. In general, however, it is not easy to demonstrate that and how governments really distribute the tariff revenues to the consumers. Furthermore, it is not easy to show if such possible transfers really are distributed to the consumers in a fair way. The tariff that maximizes the sum of the consumer surplus and tariff gain (equal to the consumer surplus in case the tariff revenue is really transferred to the consumers) is derived. The optimal tariff is presented as an explicit function of all the parameters in the demand and

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supply functions. It is also proved that the optimal tariff always is unique and strictly positive. An explicit numerical case is used to explicitly illustrate and confirm the general results.

Linear approximations of demand and supply:

The demand function in the importing nation, N_1, is found in (1). Q_D is the imported volume and P is the price. a and b are strictly positive parameters.

$$Q_D = a - bP \tag{1}$$

The export supply, Q_S, from the exporting country N_2 (or the World), is given in (2). g > 0 and h > 0.

$$Q_S = -g + hP \tag{2}$$

In equilibrium, Q_S equals Q_D, which leads to the equilibrium price (3).

$$P = \frac{a+g}{b+h} \tag{3}$$

Figure 1 illustrates the equations (1) to (3).

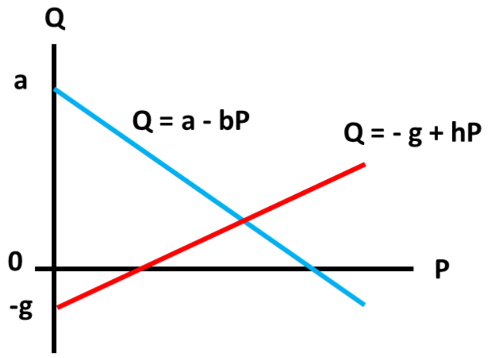

Figure 1. Illustration of the demand and supply functions. g > 0.

Since alternative price solutions will be derived in the following analysis, the free trade equilibrium price, corresponding to (3) is defined in (4).

$$P^F = \frac{a+g}{b+h} \tag{4}$$

The demand level in N_1 in free trade equilibrium, is found in (5). This can be derived via (1) and (4). This is further developed in (6).

$$Q_D^F = a - b\left(\frac{a+g}{b+h}\right) \tag{5}$$

$$Q_D^F = \frac{ah - bg}{b+h} \tag{6}$$

The export supply from N_2 (or from World) in free trade equilibrium, is found in (7). This is derived via (2) and (4). It is further developed in (8).

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$$Q_S^F = -g + h\left(\frac{a+g}{b+h}\right) \tag{7}$$

The export supply from N_2 (or from World) in free trade equilibrium is found in (8). This equals the import demand in N_1.

$$Q_S^F = \frac{ah-bg}{b+h} \quad \left(= Q_D^F\right) \tag{8}$$

The inverse demand function in N_1, (1), can be rewritten as (9). Here, the subscript is removed.

$$P = \frac{a}{b} - \frac{1}{b}Q \tag{9}$$

The inverse supply function from N_2 is found in (10).

$$P = \frac{g}{h} + \frac{1}{h}Q \tag{10}$$

The inverse demand and supply functions are illustrated in Figure 2. Strictly positive trade implies that the intersection occurs in the first quadrant. This implies that the inequality (11) holds. This is an essential observation that is needed in later derivations.

$$\frac{a}{b} > \frac{g}{h} \tag{11}$$

The consumer surplus in N_1, S, for an arbitrary value of Q, is defined in (12).

$$S = \int_0^Q \left(\frac{a}{b} - \frac{1}{b}q\right)dq - \left(\frac{g}{h} + \frac{1}{h}Q\right)Q \tag{12}$$

The consumer surplus in N1 in free trade equilibrium is found in (13). This is illustrated in Figure 3.

$$S = \int_0^Q \left(\frac{a}{b} - \frac{1}{b}q\right)dq - \left(\frac{g}{h} + \frac{1}{h}Q\right)Q \quad ,Q = Q_D^F = \frac{ah-bg}{b+h} \tag{13}$$

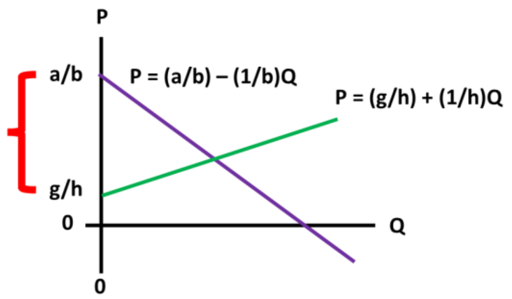

Figure 2. Inverse Demand function in N_1 and Inverse Supply function from N_2.

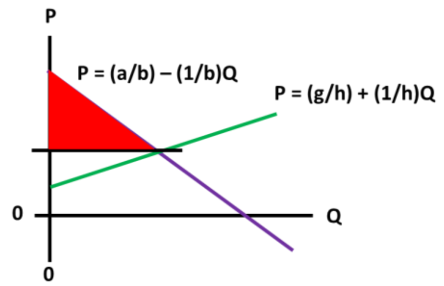

Figure 3. The consumer surplus in N_1 in free trade equilibrium. This is the area marked by red color.

Equation (13) gives (14). This is further developed to (15).

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$$S = \left[\frac{a}{b} q - \frac{1}{2b} q^2 \right]_0^Q - \frac{g}{h} Q - \frac{1}{h} Q^2 \tag{14}$$

S is a strictly concave quadratic function of Q. Furthermore, the inequality (11) implies that the coefficient of Q, in equation (15) is strictly positive. These conditions imply that there is a unique and strictly positive value of Q that maximizes S.

$$S = \left(\frac{a}{b} - \frac{g}{h} \right) Q - \left(\frac{1}{2b} + \frac{1}{h} \right) Q^2 \tag{15}$$

An explicit analysis of the consumer surplus in free trade equilibrium:

Let S^F be the consumer surplus in N_1 in free trade equilibrium. (8) and (15) give (16).

$$S^F = \left(\frac{a}{b} - \frac{g}{h} \right) \left(\frac{ah - bg}{b + h} \right) - \left(\frac{1}{2b} + \frac{1}{h} \right) \left(\frac{ah - bg}{b + h} \right)^2 \tag{16}$$

If free trade should always be desired, it should be possible to show that the consumer surplus with free trade always is the highest obtainable consumer surplus.

The following analysis investigates if some other import volume, Q, gives a larger consumer surplus than the free trade equilibrium. Figures 4 and 5 illustrate how the consumer surplus (red area) changes if the imported volume changes. When the imported volume changes, the price changes in N_1 and in the world market.

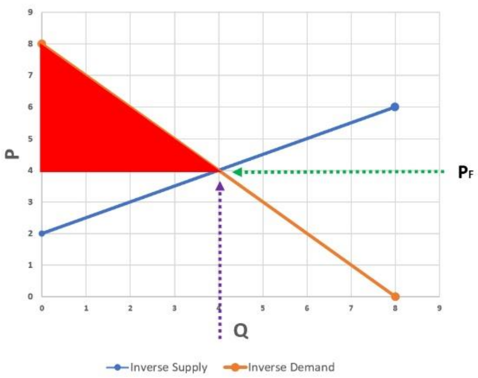

Figure 4. Illustration of the consumer surplus (red area) in free trade and P_F, the free trade price.

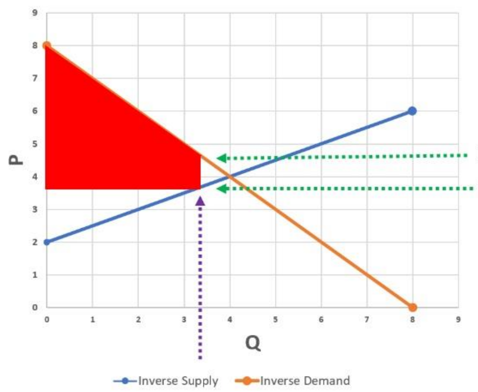

Figure 5. Illustration of the consumer surplus (red area), P_H and P_W, in case Q = 2.7.

Now, it is time for maximization of consumer surplus in N_1, via control of the total import, Q. The objective function, S, is found in equation (15).

The first order optimum condition is equation (17).

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$$\frac{dS}{dQ} = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{2}{h}\right)Q = 0 \tag{17}$$

Equation (18) is the second order maximum condition. This is always satisfied, since b and h are strictly positive.

$$\frac{d^2S}{dQ^2} = -\left(\frac{1}{b} + \frac{2}{h}\right) < 0 \tag{18}$$

Let us determine the optimal import volume from the first order optimum condition. (17) gives (19).

$$\frac{dS}{dQ} = \left(\frac{ah - bg}{bh}\right) - \left(\frac{h + 2b}{bh}\right)Q = 0 \tag{19}$$

The optimal import level is found in (20).

$$Q^* = \frac{ah - bg}{2b + h} \tag{20}$$

The optimal import volume divided by the free trade import volume is *z*. This is found in (21).

$$0 < z = \frac{Q^*}{Q^F} = \frac{\left(\frac{ah - bg}{2b + h}\right)}{\left(\frac{ah - bg}{b + h}\right)} = \frac{b + h}{2b + h} < 1 \tag{21}$$

Observation:

It is always optimal to import strictly less, than according to the free trade solution. (It is important to remember that this result comes from a partial equilibrium analysis.)

Important variables determined as functions of the optimized import volume:

Several variables may be considered as determined by the optimal import volume. One of these, equation (22), is the optimal price for the customers inside N_1, here denoted "price at home (H)". This is determined from the inverse demand function (9).

$$P_H^* = \frac{a}{b} - \frac{1}{b}Q^* \tag{22}$$

The optimal price for the exporters from the outside World (W), is found in equation (23). This is determined from the inverse supply function, (10).

$$P_W^* = \frac{g}{h} + \frac{1}{h}Q^* \tag{23}$$

The optimal tariff in equation (24) is determined as the difference between (22) and (23).

$$T^* = P_H^* - P_W^* \tag{24}$$

A deeper investigation of the optimal tariff, gives (25).

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$$T^* = \left(\frac{a}{b} - \frac{1}{b}Q^*\right) - \left(\frac{g}{h} + \frac{1}{h}Q^*\right) \tag{25}$$

The explicit function of the optimal tariff is found in (26).

$$T^* = \frac{ah - bg}{2bh + h^2} \tag{26}$$

It is possible to determine the sign of the optimal tariff. This result is surprisingly strong. Remember inequality (11).

$$(b > 0 \wedge h > 0 \wedge ah - bg > 0) \Rightarrow T^* > 0 \tag{27}$$

The optimal tariff is *always* strictly positive. This result holds, at least if other nations cannot respond with adjusted tariffs, if one nation is increasing the tariffs. In the later part of this paper, also other nations can introduce and adjust tariffs. Then, with such a problem structure, it is found that the optimal tariffs are not always strictly positive.

A concrete example:

Below, a numerically specified example is introduced to confirm and illustrate the general results reported in equations (1) to (27). The parameters are found in (28).

$$(a,b,g,h) = (8,1,4,2) \tag{28}$$

The inverse demand function is presented in (29).

$$P = 8 - Q \tag{29}$$

The inverse supply function is found in (30).

$$P = 2 + \frac{1}{2}Q \tag{30}$$

The free trade demand volume (31) and supply volume (32) are derived.

$$Q_D^F = \frac{ah - bg}{b + h} = \frac{16 - 4}{3} = 4 \tag{31}$$

$$Q_S^F = \frac{ah - bg}{b + h} = 4 \tag{32}$$

In Figure 6, the concrete free trade solution is illustrated. The consumer surplus function (15), and the parameters in equation (28) give (33) and (34).

$$S = 6Q - Q^2 \tag{33}$$

$$S = 6 \times 4 - (4)^2 = 8 \tag{34}$$

Optimization of the import level:

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Here, we optimize the import volume. A tariff is used to reduce the import volume, which also creates a tariff revenue. The objective function is the sum of the consumer surplus and the tariff revenue. This represents the total surplus of the consumers in the importing country N_1 in case the tariff revenue is redistributed to the consumers.

Observation: In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers. In such cases, it is quite possible that the consumers do not benefit from the tariff, even if the total surplus is maximized via the tariff. The optimal import volume is found in (20). With the selected parameters, we get (35).

$$Q^* = \frac{16-4}{2+2} = \frac{12}{4} = 3 \tag{35}$$

The optimal price at home, *H*, in N_1, is found in (36).

$$P_H^* = \frac{a}{b} - \frac{1}{b}Q^* = 8-3=5 \tag{36}$$

The optimal price in the World market is found in (37).

$$P_W^* = \frac{g}{h} + \frac{1}{h}Q^* = 2+\left(\frac{1}{2}\right)3 = 3.5 \tag{37}$$

The optimal tariff is presented in (38).

$$T^* = \frac{16-4}{4+2^2} = \frac{12}{8} = 1.5 \tag{38}$$

The quadratic function of the consumer surplus is found in (15). With the optimal import volume from (35), the consumer surplus becomes the value found in (39).

$$S = 6\times 3-(3)^2 = 9 \tag{39}$$

Figure 7 shows the optimal tariff solution corresponding to the concrete case based on the parameters (28). A solution, via a consumer cartel, without tariffs, is illustrated in Figure 8.

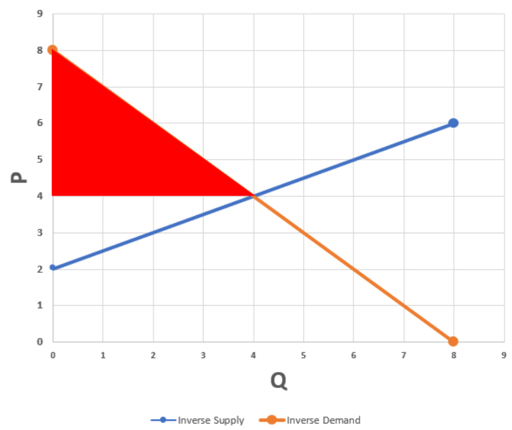

Figure 6. The concrete free trade solution. The consumer surplus is the red area.

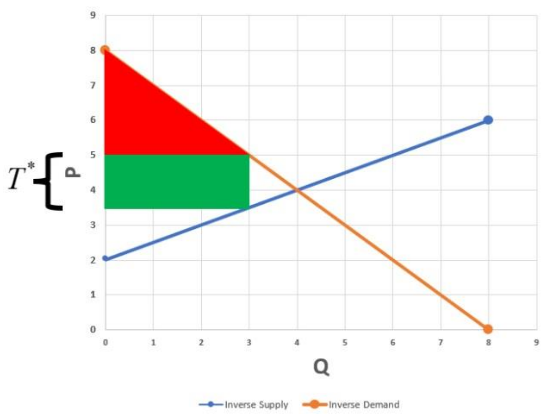

Figure 7. The concrete optimal tariff solution: The consumer surplus is the red area and the tariff gain is the green area. Consumer Surplus = Red Area =

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(3*3)/2 = 4.5; Tariff Gain = Green Area = (3*1.5) = 4.5; Total Surplus = 9.

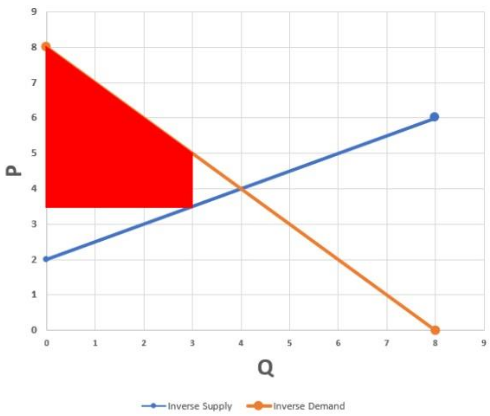

Figure 8. The consumer surplus (red area) = 9.

Some observations from the concrete example:

The total surplus in N_1 increases from 8 to 9, when the free trade solution is replaced by the optimized solution. The consumer surplus is reduced from 8 to 4.5 and the tariff revenue increases from 0 to 4.5. In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers. In such cases, it is quite possible that the consumers do not benefit from the tariff, even if the total surplus is maximized via the tariff. The free trade equilibrium price is 4, the optimal tariff is 1.5, the world price in the new equilibrium when the optimal tariff is applied is 3.5 and the price in N_1 when the optimal tariff is applied is 5. The optimal tariff is 37.5 % of the free trade equilibrium price and the optimal tariff is 42.9 % of the world price in new equilibrium when the optimal tariff is applied.

3. General equilibrium analysis.

Now, the derivations are generalized, in the sense that nonlinearities are introduced and that the prices are endogenously determined. Furthermore, more than one nation can make decisions that affect the solutions and results for all involved parties. First, the free trade solution is derived. This represents the international trade equilibrium based on optimized production and optimized consumption.

Figure 9 illustrates a typical utility function in one nation. The utility is a function of the consumption levels, c_1 and c_2, of two kinds of products, 1 and 2.

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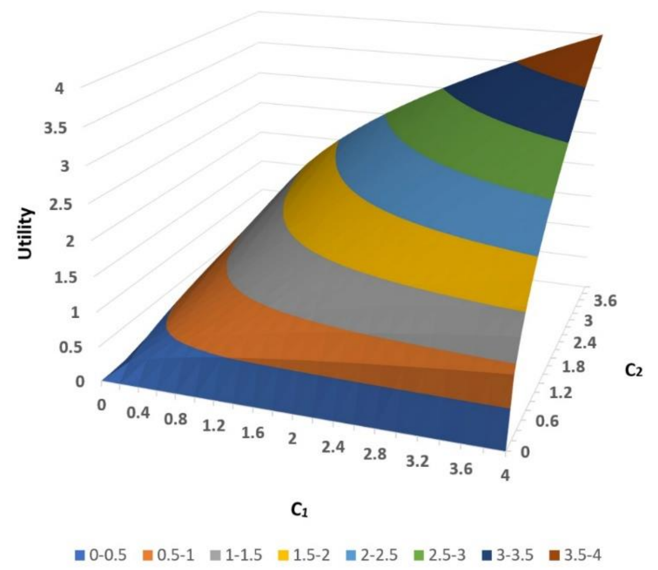

Figure 9. Typical utility function. $U = c_1^{\alpha_1} c_2^{\alpha_2}$. $\alpha_1 = \alpha_2 = \frac{1}{2}$.

Nonlinear analysis:

The utility function U_i of nation *i*, (40) is maximized via the selection of the consumption levels, $c_{i,1}, \dots, c_{i,\eta}$.

$$\max U_i = U_i\left(c_{i,1}, ., c_{i,\eta}\right) = \kappa_i c_{i,1}^{\alpha_{i,1}} \times \dots \times c_{i,\eta}^{\alpha_{i,\eta}} \tag{40}$$

The consumption budget must be satisfied. This means that the total cost of consumption must not exceed the budget, B_i . p_j denotes price of product *j*.

$$p_1 c_{i,1} + \dots + p_\eta c_{i,\eta} \le B_i \tag{41}$$

In the illustrated model, the sum of the exponents is 1, compare (42), and $\kappa_i = 1$.

$$\sum_{j=1}^{\eta} \alpha_{i,j} = 1 \tag{42}$$

The Lagrange function of consumption optimization is found in (43).

$$L_i = \kappa_i c_{i,1}^{\alpha_{i,1}} \times \dots \times c_{i,\eta}^{\alpha_{i,\eta}} + \lambda_i \left(B_i - p_1 c_{i,1} - \dots - p_\eta c_{i,\eta}\right) \tag{43}$$

The first order optimum conditions of an interior solution are reported in (44).

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$$\begin{cases} \dfrac{dL_i}{d\lambda_i} = B_i - p_1 c_{i,1} - \ldots - p_\eta c_{i,\eta} = 0 \\ \dfrac{dL_i}{dc_{i,1}} = \dfrac{\alpha_{i,1} U_i}{c_{i,1}} - p_1 \lambda_i = 0 \\ \vdots \\ \dfrac{dL_i}{dc_{i,\eta}} = \dfrac{\alpha_{i,\eta} U_i}{c_{i,\eta}} - p_\eta \lambda_i = 0 \end{cases} \tag{44}$$

Determination of the optimal solution and the demand functions:

The optimum is derived from (44).

$$\frac{dL_i}{dc_{i,j}} = \frac{\alpha_{i,j} U_i}{c_{i,j}} - p_j \lambda_i \quad \forall i \in \{1,\ldots,I\},\, j \in \{1,\ldots,\eta\} \tag{45}$$

$$\left(\frac{dL_i}{dc_{i,j}} = 0 \right) \Rightarrow \left(\frac{\alpha_{i,j} U_i}{c_{i,j}} - p_j \lambda_i = 0 \right) \tag{46}$$

Equation (46) leads to (47) and (48).

$$\frac{\alpha_{i,j} U_i}{c_{i,j}} = p_j \lambda_i \tag{47}$$

$$c_{i,j} = \frac{\alpha_{i,j} U_i}{p_j \lambda_i} \tag{48}$$

Equation (44) gives (49).

$$B_i = \sum_{j=1}^{\eta} p_j c_{i,j} \tag{49}$$

Equations (48) and (49) lead to (50).

$$B_i = \sum_{j=1}^{\eta} p_j \left(\frac{\alpha_{i,j} U_i}{p_j \lambda_i} \right) \tag{50}$$

$$\frac{U_i}{\lambda_i} = B_i \left(\sum_{j=1}^{\eta} \alpha_{i,j} \right)^{-1} = B_i \quad for \quad \sum_{j=1}^{\eta} \alpha_{i,j} = 1 \tag{51}$$

$$c_{i,j} = \frac{B_i \alpha_{i,j}}{p_j}, \quad \forall i, j \tag{52}$$

<u>Observation:</u>

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The optimal demand (or consumption) function, in nation i, for product j, is proportional to the consumption budget and to the respective utility function exponent. It is inversely proportional to the product price.

Let us consider one of the nations and two products. Compare (52). Here, in (53), the notation is simplified.

$$c_i = B\frac{\alpha_i}{p_i} \tag{53}$$

Consider the optimal consumption levels of two products, 1 and 2, as seen in (54) and (55).

$$c_1 = B\frac{\alpha_1}{p_1} \tag{54}$$

$$c_2 = B\frac{\alpha_2}{p_2} \tag{55}$$

In (56) we find that the optimal consumption ratio is a function of the relative prices. The optimal consumption ratio may also be considered as a function of the ratio of the exponents in the utility function.

$$\frac{c_2}{c_1} = \left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{p_1}{p_2}\right) \tag{56}$$

Optimal production:

The analysis should be as general as possible. However, it is necessary to specify some function form that easily captures the essential properties of the problems. The type of function used to define the production possibility frontier (57) has properties that correspond to classical models of international trade theory.

In a nation, the production possibility frontier is:

$$m_1\left(x_1\right)^2 + m_2\left(x_2\right)^2 \le 1 \quad , \quad x_1 \ge 0, x_2 \ge 0 \tag{57}$$

The parameters m_1 and m_2 may be different in different nations. The production volumes of products 1 and 2, produced in N_1, are x_{11} and x_{12}. The production volumes of products 1 and 2, produced in N_2, are x_{21} and x_{22}.

The production possibility frontier in N_1 is (58).

$$m_{11}\left(x_{11}\right)^2 + m_{12}\left(x_{12}\right)^2 \le 1 \quad , \quad x_{11} \ge 0, x_{12} \ge 0 \tag{58}$$

The production possibility frontier in N_2 is (59).

$$m_{21}\left(x_{21}\right)^2 + m_{22}\left(x_{22}\right)^2 \le 1 \quad , \quad x_{21} \ge 0, x_{22} \ge 0 \tag{59}$$

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Production possibility frontiers:

If trade should be of interest to the different countries, they should have production possibility frontiers with different properties. Figure 10 shows how this can be obtained via the functional forms present in equations (58) and (59). In Nation N_1, we have the parameters (m_{11}, m_{12}) are (1/16, 1/4), and in Nation N_2, the parameters (m_{21}, m_{22}) have the values (1/4, 1/16).

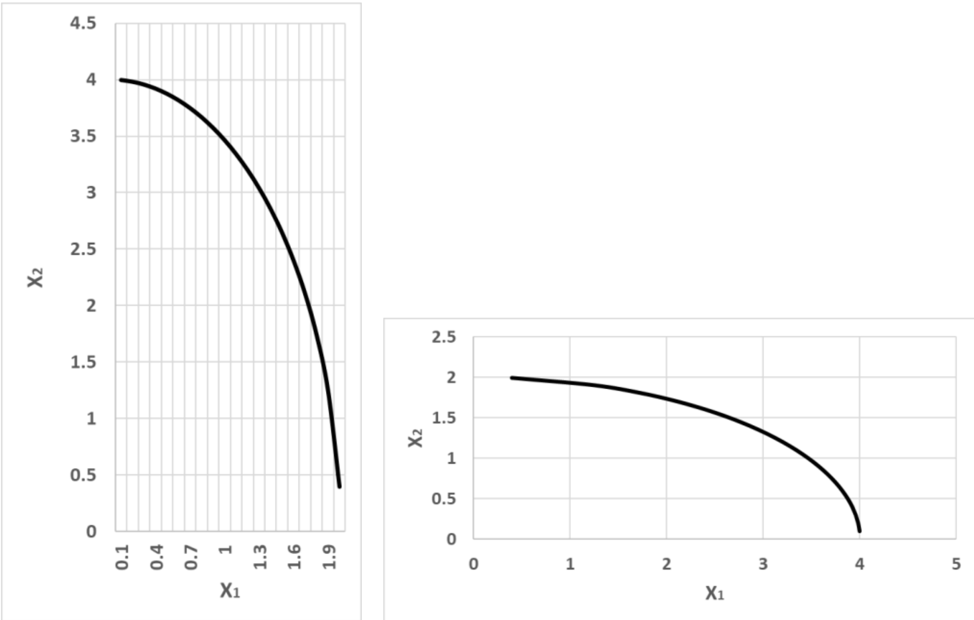

Figure 10. Production possibility frontiers in N_1 (Left) and in N_2 (right)

Production optimization:

The consumption budgets that can be used for consumption optimization, are optimized in the different nations. This is done via optimization of the production levels. The problem defined in (60) and (61), is the profit maximization problem in an arbitrary nation. The objective function π denotes the profit, which later is used as the consumption budget. In the later parts of this paper, the prices (p_1, p_2) can be defined as world market prices or prices affected by the different tariffs.

$$\max_{x_1, x_2} \pi = p_1 x_1 + p_2 x_2 \tag{60}$$

$$s.t. \quad m_1 (x_1)^2 + m_2 (x_2)^2 \leq 1 \quad , \quad x_1 \geq 0, x_2 \geq 0 \tag{61}$$

The Lagrange function of the production optimization problem is (62).

$$L = p_1 x_1 + p_2 x_2 + \lambda \left(1 - m_1 x_1^2 - m_2 x_2^2\right) \tag{62}$$

The Karush- Kuhn- Tucker conditions are applied. With full production according to the production possibility frontier, and strictly positive production of all kinds of products, the equations (63) are satisfied.

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$$\begin{cases} \dfrac{dL}{d\lambda} = 1 - m_1 x_1^{\ 2} - m_2 x_2^{\ 2} = 0 \\ \dfrac{dL}{dx_1} = p_1 - 2m_1\lambda x_1 \qquad = 0 \\ \dfrac{dL}{dx_2} = p_2 - 2m_2\lambda x_2 \qquad = 0 \end{cases} \tag{63}$$

From (63) we get (64), which leads to (65).

$$\begin{cases} x_1 = \dfrac{p_1}{2m_1\lambda} \\ x_2 = \dfrac{p_2}{2m_2\lambda} \end{cases} \tag{64}$$

$$\frac{x_2}{x_1} = \frac{m_1}{m_2}\left(\frac{p_2}{p_1}\right) \tag{65}$$

In (66), the relative price is defined.

$$Let\ R = \frac{p_1}{p_2} \tag{66}$$

The optimal production ratio is a function of the relative prices, as shown in (67).

$$\frac{x_2}{x_1} = \frac{m_1}{m_2} R^{-1} \tag{67}$$

Optimal production ratios in the two nations:

Now, the concrete numerical values of the production possibility functions in nations N_1 and N_2 are used. Compare Figure 10.

In Nation N_1, we have equation (68) and obtain the function (69).

$$(\max x_1, \max x_2) = (2,4) \Rightarrow (m_1, m_2) = \left(\frac{1}{4}, \frac{1}{16}\right) \tag{68}$$

$$\frac{x_2}{x_1} = 4R^{-1} \tag{69}$$

In Nation N_2, we have equation (70) and obtain the function (71).

$$(\max x_1, \max x_2) = (4,2) \Rightarrow (m_1, m_2) = \left(\frac{1}{16}, \frac{1}{4}\right) \tag{70}$$

$$\frac{x_2}{x_1} = \frac{1}{4} R^{-1} \tag{71}$$

Figure 11 shows how the optimal consumption ratio function and the optimal production ratio functions can be combined to determine the optimal combinations of relative prices and relative production volumes in the two nations without trade.

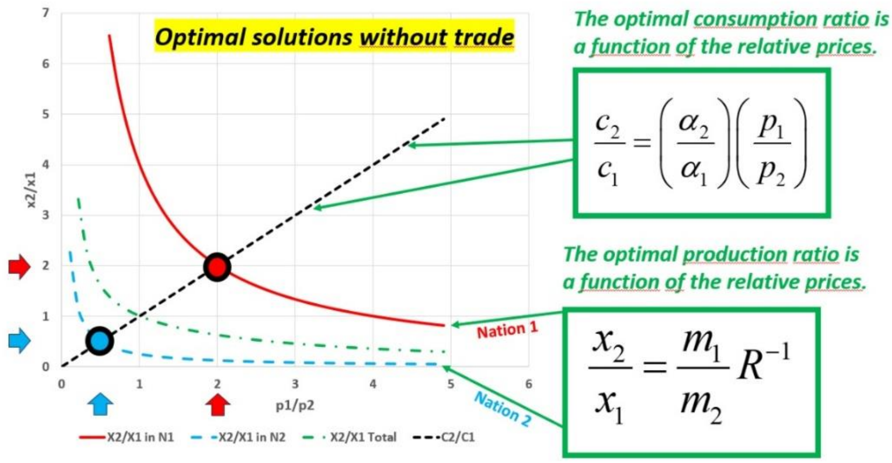

Figure 11. In the absence of international trade, the optimal combinations of production and consumption ratios, and the relative prices, can be determined via the graph.

Figure 12 shows how the optimal consumption ratio function and the optimal production ratio functions can be combined to determine the optimal combinations of relative prices and relative production volumes in the two nations with trade.

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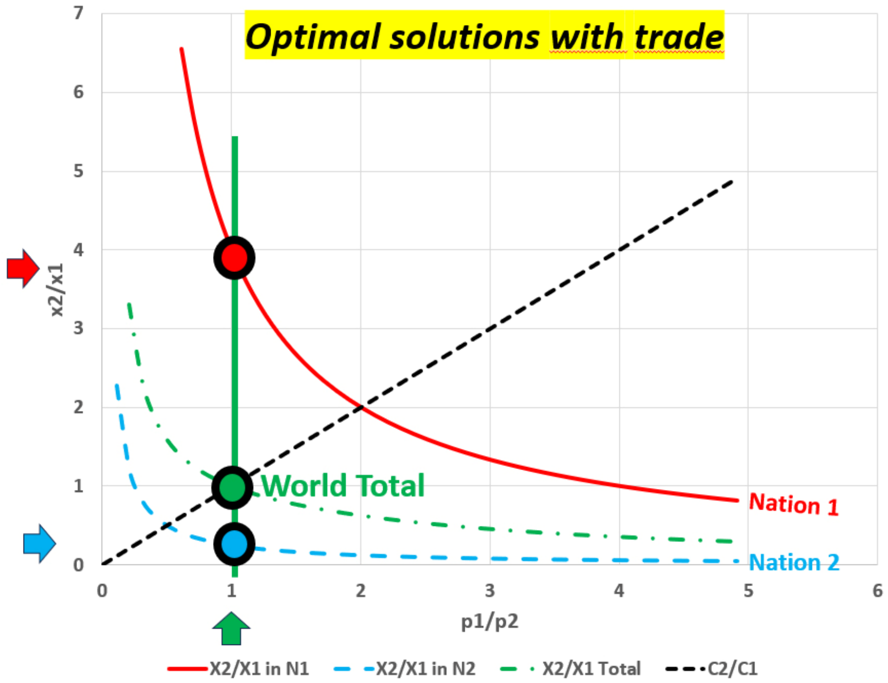

Figure 12. The equilibrium relative price in the world, with free trade, is found via the point denoted by ”World Total”. This equilibrium relative price, which is ”1” in the example, determines the optimal relative production levels in the different nations. ”x_2/x_1” increases in N_1, Nation 1, and decreases in N_2, Nation 2, when free trade is introduced. If free trade is introduced, the relative prices in the two countries converge to the same value. The optimal absolute production levels, with and without free trade, in the different nations, are graphically determined in Figures 13 and 14, based on the optimal relative production volumes and the production possibility frontiers.

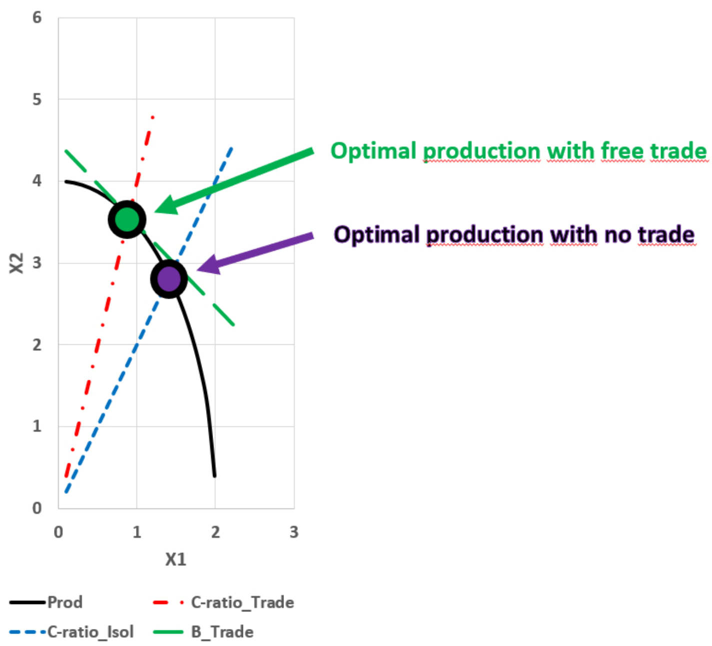

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Figure 13. Nation N_1: Free trade leads to increased specialization.

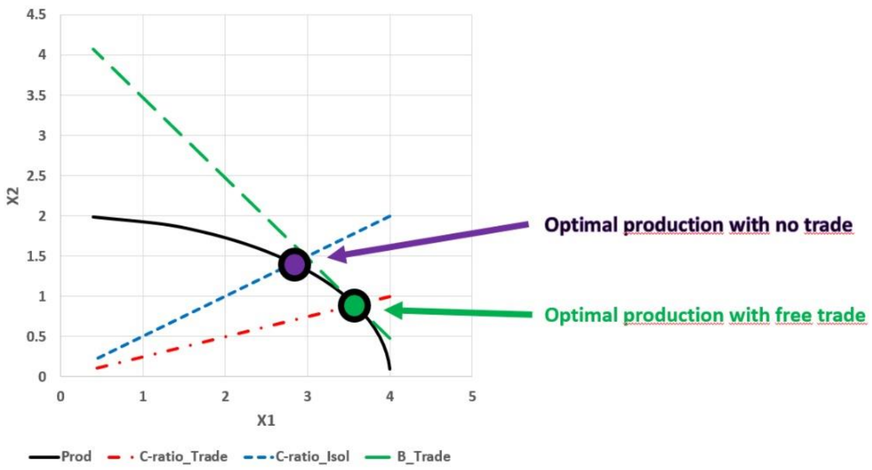

Figure 14. Nation N_2: Free trade leads to increased specialization.

Mathematical determination of the optimal absolute production volumes in the different countries:

In N_1, the optimal production volumes of the different products are determined in (72) to (79).

The production possibility frontier is (72) and the optimal product ratio is (73).

$$m_1 (x_1)^2 + m_2 (x_2)^2 = 1 \tag{72}$$

$$k = \frac{x_2}{x_1} = \frac{m_1}{m_2} R^{-1} \tag{73}$$

(73) gives (74).

$$x_2 = k\, x_1 \tag{74}$$

(72) and (74) give (75).

$$m_1 (x_1)^2 + m_2 (k\, x_1)^2 = 1 \tag{75}$$

The optimal absolute production volume of product 1 in N_1 is (76).

$$x_1 = \left(m_1 + m_2 k^2\right)^{-\left(\frac{1}{2}\right)} \tag{76}$$

Equation (73) gives (77).

$$x_1 = k^{-1} x_2 \tag{77}$$

(72) and (77) lead to (78).

$$m_1 \left(k^{-1} x_2\right)^2 + m_2 (x_2)^2 = 1 \tag{78}$$

The optimal absolute production volume of product 2 in N_1 is (79).

$$x_2 = \left(m_1 k^{-2} + m_2\right)^{-\left(\frac{1}{2}\right)} \tag{79}$$

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In nation N_2, the optimal production volumes of the different products are determined via the same logic as according to (72) to (79). Figures 15 and 16 illustrate how the optimal production in the two nations are affected by the relative price.

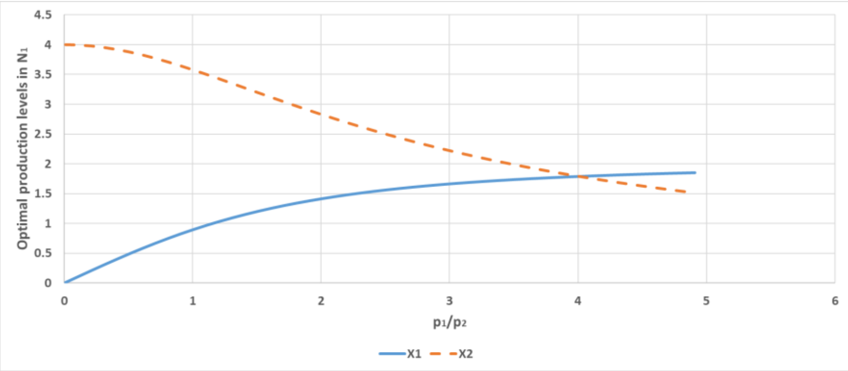

Figure 15. The optimal production levels in N_1 as a function of the relative price.

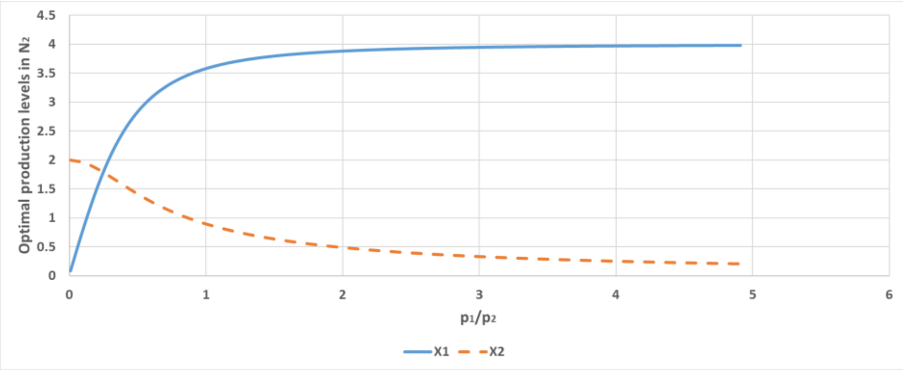

Figure 16. The optimal production levels in N_2 as a function of the relative price.

Table 1 shows the optimized consumption levels in the different nations, with free trade and without trade.

Table 1. Consumption levels in the optimized system in equilibrium.

	Nation 1 No trade	Nation 1 Free trade	Nation 2 No trade	Nation 2 Free trade
c_1	1.414	2.236	2.828	2.236
c_2	2.828	2.236	1.414	2.236

Table 2 shows the optimized consumption utility levels in the different nations, with free trade and without trade. In both nations, the utility levels are strictly higher with free trade than without trade. Observation: If the utility function exponents are high (2.0), then the ratio of the value of the utility function with free trade divided by the value of the utility function without trade, is higher than if the utility function exponents are low (0.5).

Table 2. Utility levels in the optimized system in equilibrium.

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	Nations 1 and 2, No trade Utility function exponents = (0.5, 0.5)	Nations 1 and 2, Free trade Utility function exponents = (0.5, 0.5)	Nations 1 and 2, No trade Utility function exponents = (2.0, 2.0)	Nations 1 and 2, Free trade Utility function exponents = (2.0, 2.0)
u_1	2.000	2.236	16	25
u_2	2.000	2.236	16	25

Observations of the utility levels and trade:

The utility levels in the optimized system equilibrium increase when free trade is introduced. The relative utility increase is sensitive to the values of the exponents in the utility functions. If these exponents are low, (= 0.5), then the utility levels increase by 11.80 % when free trade is introduced. If these exponents are high, (= 2.0), then the utility levels increase by 56.25 % when free trade is introduced.

4. The optimal tariffs without and with cooperation in general equilibrium

In this section, the trade solution with tariffs is studied. The trade equilibrium is determined based on optimized production and optimized consumption. One or both nations introduce tariffs on imports. Dynamic, nonlinear, non-zero sum, game theory is applied. The Nash equilibrium and the cooperative solution are determined. The optimal tariffs and utility changes in both nations, with and without cooperation, are derived.

For each relative tariff combination (T_1, T_2):

The consumption levels are functions of the consumption budgets in N_1 and N_2. The consumption budgets in N_1 and N_2 are functions of the tariff revenues. The tariff revenues are functions of the import volumes. The import volumes are functions of the consumption levels. For this reason, fix point iteration is used to determine the equilibrium. This is done via iteration in a loop, in the software, found in the appendix. Iterations with 100 steps give solutions very close to the equilibria.

For each combination (T_1, T_2):

In one loop, A, the relative prices are adjusted, until the total excess supply is zero (for every product).

In a second loop, B, within A, for each relative price level, fix point iteration is used to determine approximate solutions to all production levels, consumption levels, all consumption budgets, imported volumes and exported volumes. Furthermore, the utility levels in the nations N_1 and N_2 are determined as functions of the optimized consumption levels. W_1 and W_2 are transformed utility functions in N_1 and N_2. The transformed utility functions make it easy to investigate how utilities change when tariffs are adjusted. In Tables 3 and 4, the values of W_1 and W_2 are reported, as functions of the tariff combinations (T_1, T_2). You get the value of W_1, if you take the utility u_1 in N_1 for the tariff combination (T_1, T_2) minus the utility u_1 in N_1 for zero tariffs, and multiply this difference by 10000. W_2 corresponds to W_1, but for nation N_2.

Table 3. Numerical approximation of the transformed utility function W_1 in N_1.

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$W_1(T_1, T_2) = (u_1(T_1, T_2) - u_1(0,0)) * 10000$.

T_1 (%)	T_2(%)					
	0	20	40	60	80	100
0	0	-579	-1017	-1351	-1602	-1797
20	504	-158	-661	-1056	-1358	-1605
40	750	18	-551	-991	-1346	-1631
60	818	22	-599	-1086	-1475	-1793
80	765	-89	-751	-1280	-1704	-2050
100	629	-278	-982	-1542	-1989	-2361

Table 4. Numerical approximation of the transformed utility function W_2 in N_2.

$W_2(T_1, T_2) = (u_2(T_1, T_2) - u_2(0,0)) * 10000$.

T_1 (%)	T_2 (%)					
	0	20	40	60	80	100
0	0	501	748	825	766	626
20	-581	-158	13	23	-93	-275
40	-1018	-666	-551	-602	-754	-979
60	-1348	-1055	-993	-1086	-1283	-1543
80	-1602	-1360	-1348	-1477	-1704	-1990
100	-1798	-1603	-1629	-1794	-2051	-2361

Determination of an approximation $W_1(T_1, T_2)$:

Hypothesis: The function can be approximated by a multivariate polynomial of the second order. This is found in (80). This may be considered as a second order Taylor approximation.

$$W_1(T_2, T_2) = k_1T_1 + k_{11}T_1^2 + k_2T_2 + k_{12}T_1T_2 + k_{22}T_2^2 \tag{80}$$

Regression analysis based on the numerically determined values of $W_1(T_1, T_2)$:

When the hypothesis (80) and the observations reported in Table 3 are used as raw data, the estimated function (81) is obtained:

$$W_1 \approx 2399\,T_1 - 1820\,T_1^2 - 3102\,T_2 - 1279\,T_1T_2 + 1349\,T_2^2 \tag{81}$$

More details about the regression analysis are reported in Tables 5 and 6. In Table 5 we note that the value of R is very close to 1. Table 6 shows that all the P-values are extremely low. For these reasons,

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we know that the estimated function (81) almost perfectly replicates the data in Table 3. This is very important, since the continued derivations are based on the approximating function (80) and the numerically specified version (81).

Table 5. Regression statistics part 1.

Multiple R	0.999251068
R Square	0.998502696
Adjusted R Square	0.966051431
Standard Error	50.31183919
Observations	36

Table 6. Regression statistics part 2.

	Coefficients	Standard Error	t Stat	P-value
Intercept	0	---	---	---
T_1	2399	72.00	33.32	7.684×10^{-26}
$T_1^{\ 2}$	- 1820	78.26	- 23.26	3.478×10^{-21}
T_2	- 3102	72.00	- 43.08	3.153×10^{-29}
T_1T_2	- 1279	58.33	- 21.93	1.936×10^{-20}
$T_2^{\ 2}$	1349	78.26	17.24	1.945×10^{-17}

Figure 17 shows the transformed utility function W_1 in nation N_1. In the yellow region, if T_2 is zero, N_1 benefits the most if T_1 is approximately 66%. For large values of T_2, W_1 is strictly negative, for all values of T_1. It is also clear that, for every possible value of T_2, there is an optimal value of T_1. As T_2 increases, the optimal value of T_1 decreases. We may imagine the W_1 function as a mountain, with a mountain ridge. One part of the ridge is located above the ocean level, this is where T_2 is low. At the ocean level, the value of W_1 is zero. For high values of T_2, the mountain ridge is located below the surface of the ocean. The optimal value of T_1 is always found on the mountain ridge. The location is determined by the value of T_2.

The approximated function W_1 is now assumed to be exact (82).

$$W_1 = 2399\,T_1 - 1820\,T_1^2 - 3102\,T_2 - 1279\,T_1T_2 + 1349\,T_2^2 \tag{82}$$

The first derivative of (82) with respect to T_1 is (83).

$$\frac{dW_1}{dT_1} = 2399 - 3640\,T_1 - 1279\,T_2 \tag{83}$$

Since the analytical background to the problem is exactly known, and symmetry is present, (82) leads to (84) and (85).

$$W_2 = 2399\,T_2 - 1820\,T_2^2 - 3102\,T_1 - 1279\,T_1T_2 + 1349\,T_1^2 \tag{84}$$

$$\frac{dW_2}{dT_2} = 2399 - 3640\,T_2 - 1279\,T_1 \tag{85}$$

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The first derivative of W_1 with respect to T_1 as a function of T_1 and T_2 is illustrated in Figure 18 in three dimensions. Compare (83). The graph clearly shows the region of T_1 and T_2 where it is optimal for nation N_1 to increase the value of T_1 and where it is optimal to decrease T_1.

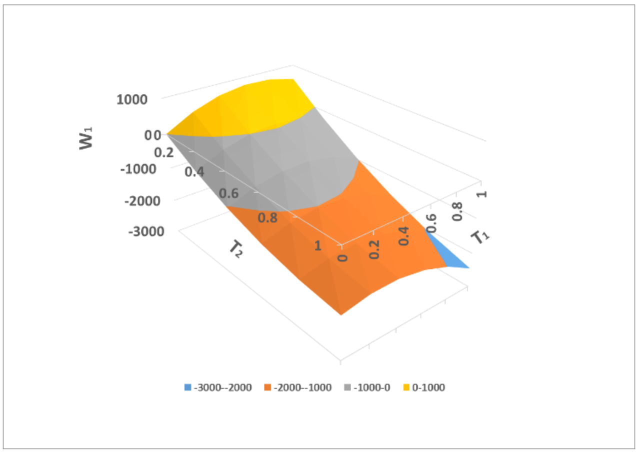

Figure 17. Illustration of W_1, the transformed utility function in N_1, via the approximated function (81).

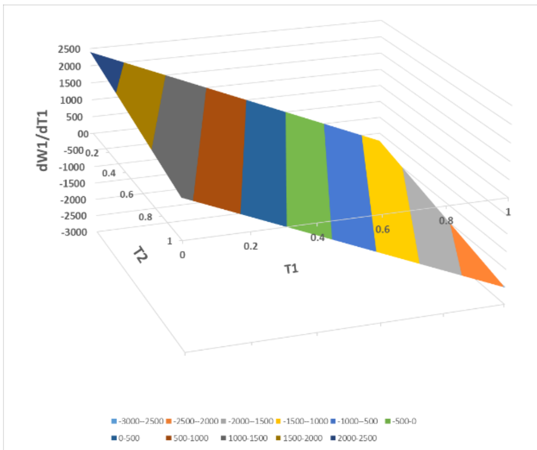

Figure 18. Illustration of the first derivative of W_1 with respect to T_1 as a function of T_1 and T_2 in three dimensions. Compare (83).

Basic principles:

N_1 maximizes W_1 via the selection of T_1. If the derivative (83) exceeds 0, it is rational to increase T_1. If $T_2 = 0$, it is always rational for N_1 to increase T_1 from zero. Then, the optimal value of T_1 is determined according to equation (86).

$$\left(\frac{dW_1}{dT_1} = 2399 - 3640\,T_1 = 0\right) \Rightarrow T_1 \approx 66\% \tag{86}$$

Hence, N_1 finds it optimal to introduce a 66% tariff and deviate from free trade. N_2 maximizes W_2 via the selection of T_2. If the derivative (85) exceeds 0, it is rational to increase T_2. If $T_1 = 0$, it is always rational for N_2 to increase T_2 from zero. Then, the optimal value of T_2 is determined according to equation (87).

$$\left(\frac{dW_2}{dT_2} = 2399 - 3640\,T_2 = 0\right) \Rightarrow T_2 \approx 66\% \tag{87}$$

Here, N_2 finds it optimal to introduce a 66% tariff and deviate from free trade.

Nash equilibrium:

N_1 maximizes W_1 via T_1, conditional on the fact that N_2 maximizes W_2 via T_2. See (88).

$$\frac{dW_1}{dT_1} = 2399 - 3640\,T_1 - 1279\,T_2 = 0 \tag{88}$$

N_2 maximizes W_2 via T_2, conditional on the fact that N_1 maximizes W_1 via T_1. Compare (89).

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$$\frac{dW_2}{dT_2} = 2399 - 1279\,T_1 - 3640\,T_2 = 0 \tag{89}$$

The equation system (90) is satisfied in the unique Nash equilibrium:

$$\begin{bmatrix} 3640 & 1279 \\ 1279 & 3640 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 2399 \\ 2399 \end{bmatrix} \tag{90}$$

Cramer's rule and (90) gives the explicit Nash equilibrium tariff solutions (91) and (92).

$$T_1 = \frac{\begin{vmatrix} 2399 & 1279 \\ 2399 & 3640 \end{vmatrix}}{\begin{vmatrix} 3640 & 1279 \\ 1279 & 3640 \end{vmatrix}} \approx \frac{5664039}{11613759} \approx 0.48770 \approx 49\% \tag{91}$$

$$T_2 = \frac{\begin{vmatrix} 3640 & 2399 \\ 1279 & 2399 \end{vmatrix}}{\begin{vmatrix} 3640 & 1279 \\ 1279 & 3640 \end{vmatrix}} \approx \frac{5664039}{11613759} \approx 0.48770 \approx 49\% \tag{92}$$

Figure 19 illustrates the derivations that lead to the Nash equilibrium solution. Figure 20 shows how the nations N_1 and N_2 should adjust the values of T_1 and T_2 respectively, to optimize the respective objective functions W_1 and W_2, in different situations. Blue arrows correspond to optimal changes of T_1 and red arrows correspond to optimal changes of T_2.

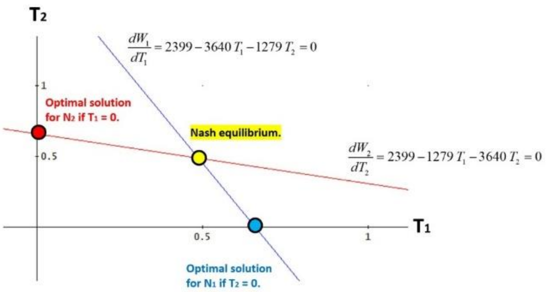

Figure 19. Illustration of the Nash solution.

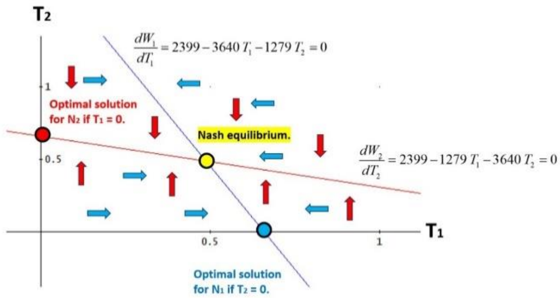

Figure 20. Directions of optimal tariff changes.

Optimal dynamic changes of the tariffs are illustrated in Figures 21 to 23, provided that the system starts with free trade without any tariffs. The first move is taken by nation N_1, as shown in Figure 21.

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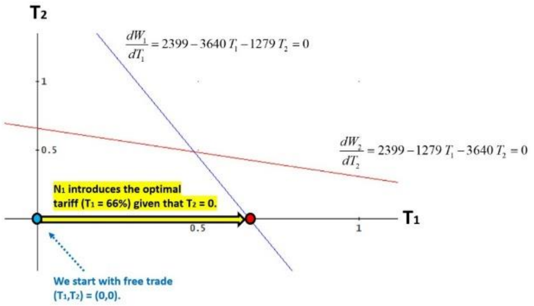

Figure 21. Optimal tariff adjustment 1.

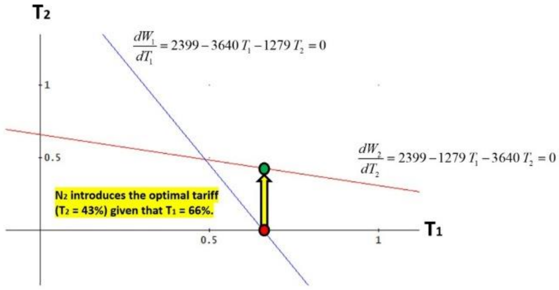

Figure 22. Optimal tariff adjustment 2.

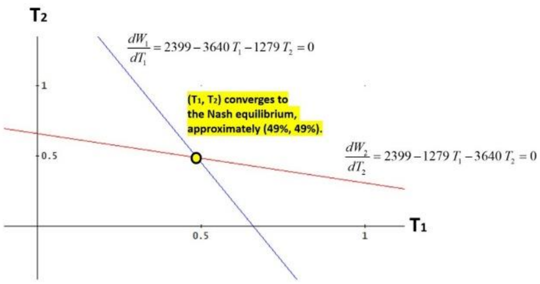

Figure 23. The Tariffs converge to the Nash equilibrium.

The Nash equilibrium tariff combination is determined from the equation system in (90). The system is simplified in (93) where all elements from (90) are divided by 2399.

$$\begin{bmatrix} 1.517 & 0.533 \\ 0.533 & 1.517 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{93}$$

The system (93) is generalized to (94).

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{94}$$

The Nash equilibrium values of the tariffs are derived via the general functions and the approximate parameters, in equations (95) and (96).

$$T_1 = \frac{\begin{vmatrix} 1 & \alpha_{12} \\ 1 & \alpha_{22} \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}} = \frac{\alpha_{22} - \alpha_{12}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \qquad \approx \frac{1.517 - 0.533}{(1.517)^2 - (0.533)^2} \approx 0.488 \tag{95}$$

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$$T_2 = \frac{\begin{vmatrix} \alpha_{11} & 1 \\ \alpha_{21} & 1 \end{vmatrix}}{\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}} = \frac{\alpha_{11} - \alpha_{21}}{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}} \approx \frac{1.517 - 0.533}{(1.517)^2 - (0.533)^2} \approx 0.488 \tag{96}$$

It is important to know if the tariff levels will converge to the Nash equilibrium. In the investigation of the dynamics of the tariff system, the notation is simplified according to (97).

$$\left(T_1, T_2\right) = (m, n) \tag{97}$$

From (94) and (97), the system (98) is obtained.

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{98}$$

Tariff Time Loop:

It is assumed that the system starts in free trade, where all tariffs are zero. Nation 1, N_1, starts the deviation from free trade by increasing T_1, denoted m, to the optimal value, conditional on the value of T_2, denoted n. In the iterations, the index t is applied.

m_t is calculated based on n_t, as shown in (99), and n_{t+1} is calculated based on m_t, via (100).

$$\left(\alpha_{11} m + \alpha_{12} n = 1\right) \Rightarrow m_t = \frac{1}{\alpha_{11}}\left(1 - \alpha_{12} n_t\right) \tag{99}$$

$$\left(\alpha_{21} m + \alpha_{22} n = 1\right) \Rightarrow n_{t+1} = \frac{1}{\alpha_{22}}\left(1 - \alpha_{21} m_t\right) \tag{100}$$

With a step in time, (100) gives (101).

$$n_t = \frac{1}{\alpha_{22}}\left(1 - \alpha_{21} m_{t-1}\right) \tag{101}$$

Equations (99) and (101) give (102), where n has been eliminated.

$$m_t = \frac{1}{\alpha_{11}}\left(1 - \alpha_{12}\left(\frac{1}{\alpha_{22}}\left(1 - \alpha_{21} m_{t-1}\right)\right)\right) \tag{102}$$

Equations (103), (104) and (105), lead to a presentation of m_t as a first order autoregressive process with two parameters, K and L, in (106).

$$m_t = \frac{1}{\alpha_{11}}\left(1 - \alpha_{12}\left(\frac{1}{\alpha_{22}} - \frac{\alpha_{21}}{\alpha_{22}} m_{t-1}\right)\right) \tag{103}$$

$$m_t = \frac{1}{\alpha_{11}}\left(1 - \frac{\alpha_{12}}{\alpha_{22}} + \frac{\alpha_{12}\alpha_{21}}{\alpha_{22}} m_{t-1}\right) \tag{104}$$

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$$m_t = \frac{\left(1-\frac{\alpha_{12}}{\alpha_{22}}\right)}{\alpha_{11}} + \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} m_{t-1} \tag{105}$$

$$m_t = K + L\,m_{t-1} \quad , \quad K = \frac{\left(1-\frac{\alpha_{12}}{\alpha_{22}}\right)}{\alpha_{11}}, L = \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} \tag{106}$$

The parameters K and L in the first order autoregressive process of m, representing T_1, determine the dynamics of the optimal tariff T_1, and indirectly of the dynamically linked optimal tariff T_2. According to (106), the parameters K and L are known functions of the parameters in the equation system (98), which means that the dynamic properties, such as convergence, are known functions of the parameters in (98). The tariff system converges to the Nash equilibrium if K > 0 and 0 < L < 1. Figure 24 illustrates the dynamics of the system (98).

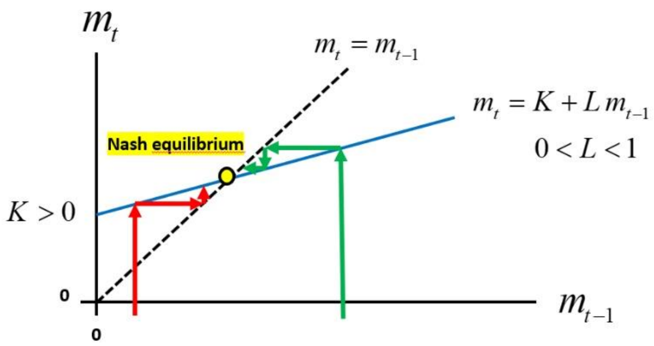

Figure 24. The tariffs converge if K > 0 and 0 < L < 1.

The tariff system converges to the Nash equilibrium if the conditions in (107) are satisfied.

$$K = \frac{\left(1-\frac{\alpha_{12}}{\alpha_{22}}\right)}{\alpha_{11}} > 0 \wedge 0 < L = \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} < 0 \tag{107}$$

It is possible to investigate if the equation system based on the parameters in (90) leads to convergence. Equations (108) and (109) show that the parameters from (90) imply that K and L obtain values such that convergence to the Nash equilibrium is obtained.

$$K = \frac{\left(1-\frac{\alpha_{12}}{\alpha_{22}}\right)}{\alpha_{11}} = \frac{\left(1-\frac{0.533}{1.517}\right)}{1.517} \approx 0.428 > 0 \tag{108}$$

$$0 < L = \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} = \frac{0.533 \times 0.533}{1.517 \times 1.517} \approx 0.123 < 1 \tag{109}$$

The conclusion based on (108) and (109) is that the tariff system converges to the Nash equilibrium. This is also confirmed via the illustrations in Figures 21 to 23, and in Figure 24.

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The Nash equilibrium in general form via the general iteration function: From (105), we get (110). In dynamic equilibrium, the value of the process will not change over time.

$$m^{NE} = \frac{\left(1 - \frac{\alpha_{12}}{\alpha_{22}}\right)}{\alpha_{11}} + \frac{\alpha_{12}\alpha_{21}}{\alpha_{11}\alpha_{22}} m^{NE} \tag{110}$$

Equation (110) is transformed to (111).

The dynamic equilibrium value of m is found in (111) as a function of the system parameters. This is identical to the equilibrium value of T_1 according to the earlier calculation. Compare equation (95).

$$m^{NE} = \frac{(\alpha_{22} - \alpha_{12})}{(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})} \tag{111}$$

Optimal cooperation:

Tables 3 and 4 and Figure 20 show that, starting with free trade, both nations prefer to increase the respective tariffs, provided that the other nations do not increase the tariffs. In case the different nations adjust the tariff levels independently and optimally, based on the tariffs presented by the other nation(s), the tariffs converge to the Nash tariff equilibrium. However, it is not necessarily the case that the Nash equilibrium is a better solution for the different nations, than the original solution, with free trade and no tariffs. Table 7 clearly shows that the sum of the transformed utility functions decreases if T_1, T_2 or both T_1 and T_2 increase.

Table 7. Numerical approximation of the sum of the transformed utility functions in all nations, $W_{SUM}(T_1, T_2) = W_1(T_1, T_2) + W_2(T_1, T_2)$. Compare Tables 3 and 4. Observations: The maximum of the total utility is obtained when all tariffs are zero. The total utility decreases if some tariff(s) increase(s).

T_1 (%)	T_2 (%)					
	0	20	40	60	80	100
0	0	-78	-269	-527	-836	-1171
20	-77	-316	-648	-1033	-1451	-1880
40	-268	-647	-1101	-1593	-2100	-2610
60	-530	-1033	-1592	-2173	-2758	-3336
80	-837	-1450	-2099	-2757	-3407	-4041
100	-1169	-1881	-2612	-3336	-4041	-4721

The derived Nash equilibrium from (95) and (96) gives (112).

$$(T_1, T_2) = (0.48770, 0.48770) \tag{112}$$

Based on the Nash equilibrium tariffs (95) and (96), and the transformed utility functions (80) and (84), equation (113) is derived. Clearly, both nations get a worse situation in the Nash equilibrium, than with free trade, without tariffs.

$$W_1 = W_2 \approx -759 \tag{113}$$

Observation: It is better for N_1 and for N_2 to select free trade, without any tariffs, than to fight a tariff war, and select the best possible tariff levels, based on the tariffs in the other nations. If both nations play optimally according to the Nash equilibrium, they both gain if they cooperate and reduce all tariffs to zero. Then, free trade is obtained. This free trade is however unstable. Some nation may stop the cooperation, increase tariffs, and assume or hope that the other nations will become too scared to respond with the optimal tariff changes. If the other nations respond optimally, the Nash solution will be found. Then, again, all rational nations will find it rational to cooperate and move back to the free trade solution without tariffs.

5. Conclusions

Optimal tariff strategies—both with and without strategic responses—are derived using two analytical frameworks: a linear partial equilibrium trade model (Model A) and a nonlinear general equilibrium trade model encompassing two nations (Model B). Under conditions of free trade, both models demonstrate that it is always advantageous for one nation (denoted N_1) to impose a strictly positive tariff on imports (T_1) provided that other nations maintain zero tariffs. Within the framework of Model B, it is further shown that if N_1 introduces a tariff, it becomes rational for the second nation (N_2) to respond by imposing its own tariff (T_2). The resulting Nash equilibrium in tariffs is unique and stable. Nevertheless, welfare levels in both nations are found to be higher under free trade than under the Nash equilibrium outcome with positive tariffs. When both nations anticipate the reciprocal adjustments of their trading partners and select their tariffs optimally, deviation from free trade is no longer rational. Thus, mutual free trade agreements emerge as rational and welfare-enhancing for all participating nations. However, in a non-democratic setting, a dictator may have an incentive to impose tariffs even when this policy reduces national consumer welfare, as the tariff revenues can be appropriated for alternative uses. In such cases, tariffs diminish consumer surplus in both the exporting and importing countries.

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The author declares no conflict of interest.

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Appendix

The computer code can be used to derive the numerical results reported in Tables 3, 4 and 7 in the main text. The code is developed in QB64. The algorithm is described in the main text. The code may be obtained from the author.